A Primer on Quantitative Risk Measures

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Ever since Harry Markowitz's pioneering work on portfolio construction in 1952, the measurement of portfolio risk that has been a cornerstone of investment theory and practice is *variance* or its square root, *standard deviation*.[1] While Markowitz used variance as the measure of risk in his original model, over the past few decades, a number of researchers, including Markowitz himself, have proposed alternative risk measures. In this article, I explain these various risk measures, their motivation, and how some of them are used in measures of risk-adjusted performance.

Variance and Expected Utility Theory

The problem of constructing an investment portfolio is an example of a class of problems involving making decisions under uncertainty, i.e., problems in which someone has to make decisions today which effect outcomes that cannot be known until sometime in the future. In the 1940s, John von Neumann and Oskar Morgenstern developed a framework for developing models of decision making under certainty known as *expected utility theory*.[2] Expected utility theory had a major impact on Harry Markowitz's approach to his theory of portfolio construction.[3]

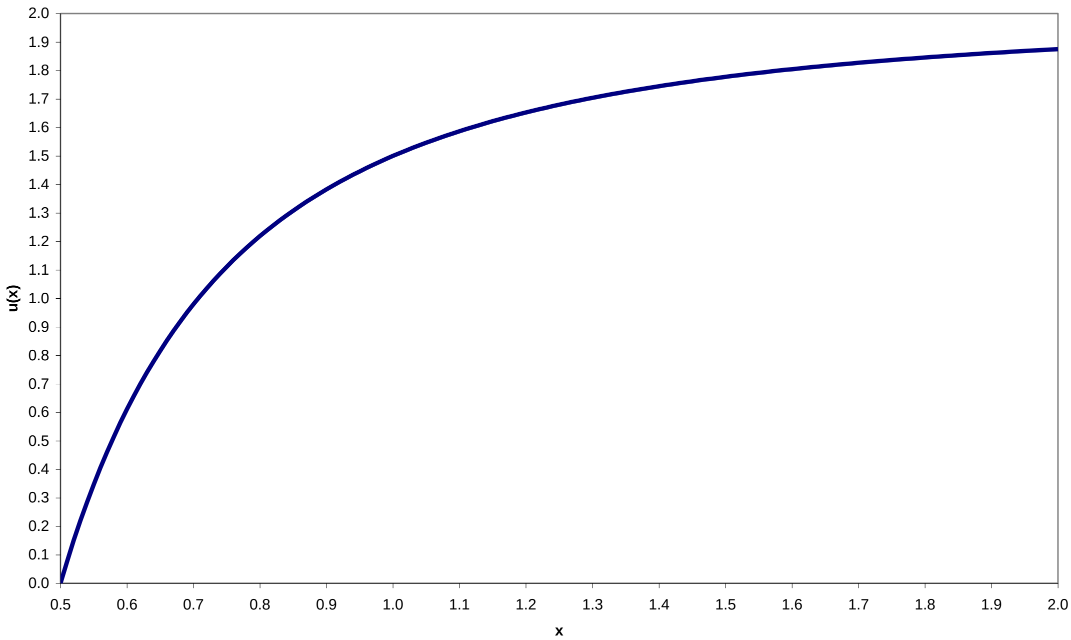
According to expected utility theory, a decision maker's attitudes towards risk can be described by a *utility function* of some future quantity that the decision is concerned about such as consumption or wealth. As Figure 1 illustrates, the utility function is assumed to be increasing and concave; the former because the decision maker prefers more to less of the quantity in question; the latter because the decision maker is assumed to be risk averse.

[1] Markowitz, Harry M., "Portfolio Selection," *Journal of Finance*, Vol. 7, Issue 1, pp. 77–91, 1952.

[2] Neumann, John von and Oskar Morgenstern *Theory of Games and Economic Behavior*. Princeton, NJ. Princeton University Press. 1944, second .ed. 1947, third .ed. 1953.

[3] Sam Savage recalls that when he met Harry Markowitz for the first time, "He [Markowitz] told me he had been indoctrinated at point-blank range in expected utility theory by my dad [Leonard J. Savage]." (Markowitz, Harry M., Sam Savage, and Paul D. Kaplan, "What Does Harry Markowitz Think?" *Morningstar Advisor*, June/July 2010.)

Figure 1: A Von Neumann-Morgenstern Utility Function



By saying that decision makers are risk averse, we mean that they always prefer a certain outcome to an uncertain outcome that has the same expected value. In other words, if X is a random variable representing the uncertain quantity that a decision maker is concerned about, receiving E[X] with certainty is always preferred to receiving X.

Under the assumptions of expected utility theory, the decision maker ranks alternatives by the expected value of the utility function applied to the quantity in question. Letting u(.) denote the utility function, risk aversion implies that

$$\mathrm{u}\left(\mathrm{E}\left[X\right]\right) \geq \mathrm{E}\left[\mathrm{u}\left(X\right)\right] \tag{1}$$

Since we have assume that u(.) is concave, *Jensen's Inequality* implies that inequality (1) must hold.

In his 1959 book, Markowitz explains the principles of expected utility theory and attempts to use it as rationalization for the mean-variance model that first presented in 1952.[4] However, he did not fully achieve a full rationalization until twenty years later in a paper he co-authored with Haim Levy.[5]

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[4] Markowitz, Harry .M. *Portfolio Selection: Efficient Diversification of Investments*, New York: John Wiley & Sons, 1959.

[5] Levy, Haim and Harry M. Markowitz, "Approximating Expected Utility by a Function of Mean and Variance," *American Economic Review*, June1979.

Levy and Markowitz developed an approximation for expected utility based on a Taylor series expansion. Suppose that u(.) is a twice differentiable von Neumann-Morgenstern utility function. Suppose that the decision maker is an investor who has invested one unit of money into a portfolio that must be constructed today. Let $\tilde{r}_P$ be a random variable that will equal the rate of return of a given portfolio p. The investor ranks alternative portfolios by their respective expected utilities.

Levy and Markowitz consider the second-order Taylor series expansion of $u(1+\tilde{r}_P)$ around $1+E[\tilde{r}_P]$ which is

$$u(1+\tilde{r}_P) \approx u\left(1+E[\tilde{r}_P]\right) + u'\left(1+E[\tilde{r}_P]\right)\left(\tilde{r}_P - E[\tilde{r}_P]\right) + \tfrac{1}{2}u''\left(1+E[\tilde{r}_P]\right)\left(\tilde{r}_P - E[\tilde{r}_P]\right)^2 \qquad (2)$$

Since the variance of $\tilde{r}_P$ is

$$\sigma^2[\tilde{r}_P] = E\left[\left(\tilde{r}_P - E[\tilde{r}_P]\right)^2\right] \qquad (3)$$

It follows that $E\left[u(1+\tilde{r}_P)\right]$ can be approximated as follows:

$$E\left[u(1+\tilde{r}_P)\right] \approx u\left(1+E[\tilde{r}_P]\right) + \tfrac{1}{2}u''\left(1+E[\tilde{r}_P]\right)\sigma^2[\tilde{r}_P] \qquad (4)$$

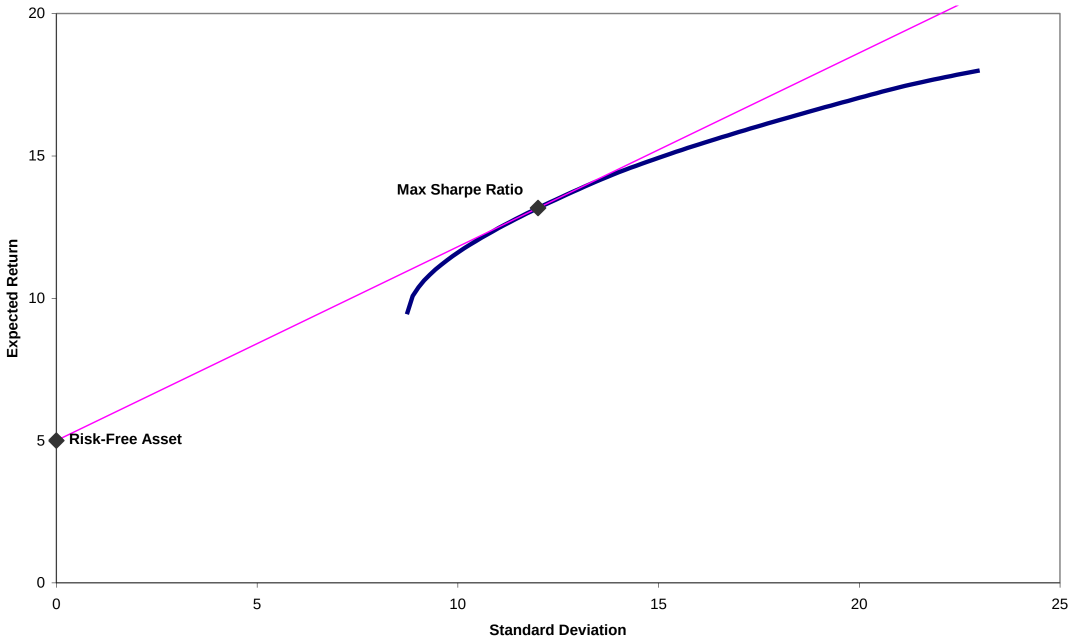
Since u(.) is concave, u”(.) is negative. Hence equation (4) shows that an expected utility maximizing investor would be well served by limiting portfolio choices to those that have the highest possible expected return for any given level of variance or standard deviation. In other words, a reasonable approximation to rational portfolio choice is to consider portfolios along Markowitz’s mean-variance efficient frontier as shown in Figure 2.

Standard deviation is the most common used risk measure. In particular it is the denominator of the Sharpe ratio, which is probably the most commonly used measure of risk-adjusted performance. In *ex ante* form, the Sharpe ratio is:

$$\text{ShR}[\tilde{r}_P] = \frac{E[\tilde{r}_P] - r_f}{\sigma[\tilde{r}_P]} \qquad (5)$$

where r_F is the rate of return on a risk-free investment, such as a government treasury bill. As Figure 2 shows, an investor who seeks the portfolio with the highest possible Sharpe ratio would select a portfolio along the Markowitz efficient frontier.

Figure 2: Markowitz Frontier and Portfolio with Maximum Sharpe Ratio



Downside Risk

For an investor, risk is not merely the volatility of returns, but the possibility of losing money. This observation has led a number of researchers, including Markowitz himself in his 1959 book, to propose "downside" measures of risk as alternatives to standard deviation which only look at the part of the return distribution that is lower than either the mean or a given target[6].

W. Van Harlow defines the n^{th} "lower partial moment" for a given target rate of return, τ, as:[7]

$$\mathrm{LPM}_n\left(\tilde{r}_P;\tau\right)=\mathrm{E}\left[\max\left(\tau-\tilde{r}_P,0\right)^n\right] \tag{6}$$

In particular, $\mathrm{LPM}_2\left(\tilde{r}_P;\tau\right)$ is what Markowitz[8] and others call the *target semivariance*.

[6] Chapter 9 is entirely devoted to this topic. See note 4 for the citation.

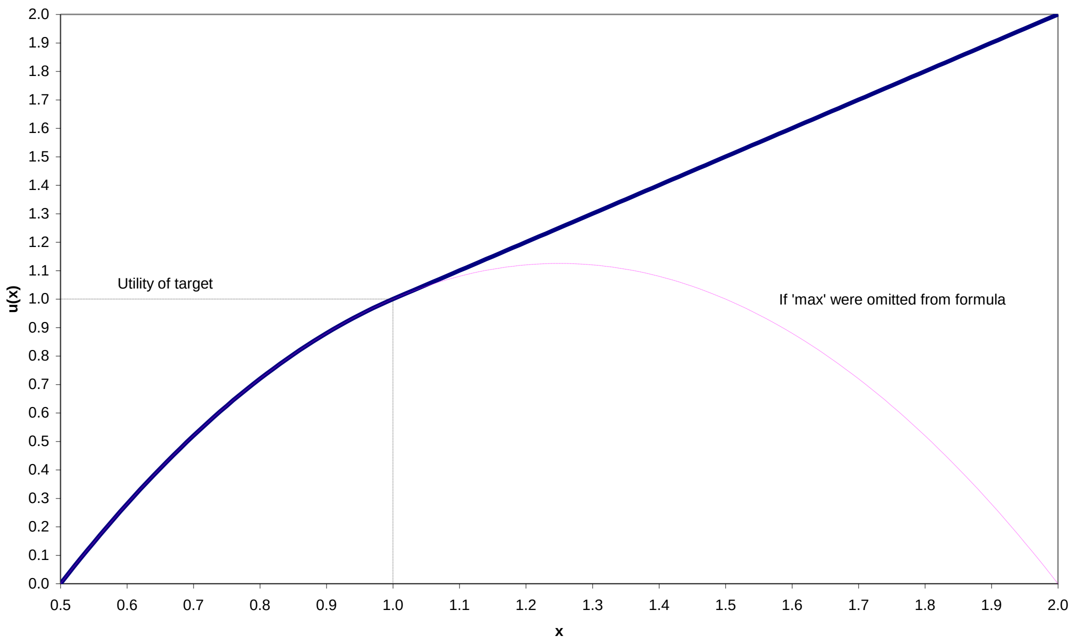
[7] Harlow, W. Van, "Asset Allocation in a Downside-Risk Framework," *Financial Analysts Journal*, September/October 1991.

Peter Fishburn showed that $\mathrm{LPM}_2(\tilde{r}_P;\tau)$ can be motivated by expected utility theory by assuming that the utility function u(.) takes the form

$$u(x) = x - k\cdot\max(1+\tau-x,0)^n \tag{7}$$

where k is a parameter for the degree of risk aversion.[9] Figure 3 shows the Fishburn utility function with k=XX and n=2.

Figure 3: A Fishburn Utility Function



If an investor's attitudes towards risk can be expressed with the Fishburn utility function given in equation (7), the expected utility of a risky portfolio is

$$E\left[u(1+\tilde{r}_P)\right] = 1 + E\left[\tilde{r}_P\right] - k\cdot\mathrm{LPM}_n\left[\tilde{r}_P;\tau\right] \tag{8}$$

Hence, an investor with a Fishburn utility function picks a portfolio on a mean-LPM frontier. The portfolio along the portfolio selected depends on the value of the parameter k.

Just as variance is often represented by its square root, standard deviation, target semivariance is often by its square root, *downside deviation* which we write as:

$$\mathrm{DD}\left[\tilde{r}_P;\tau\right] = \sqrt{\mathrm{LPM}_2\left[\tilde{r}_P;\tau\right]} \tag{9}$$

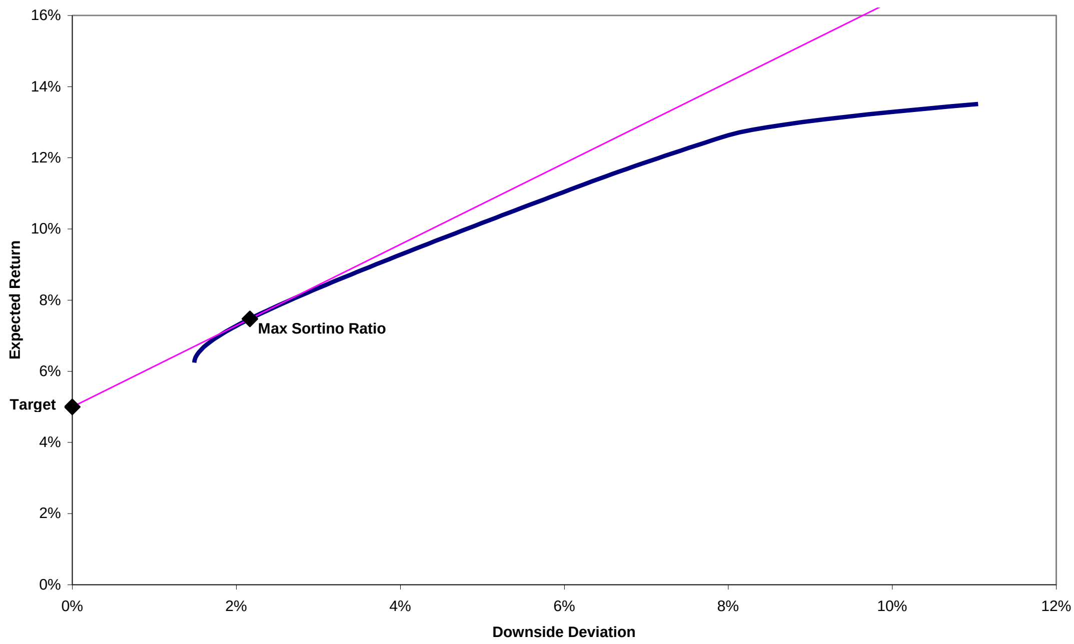
[9] Fishburn, Peter C., "Mean-Risk Analysis with Risk Associated with Below-Target Returns," *American Economic Review*, March 1977.

Frank Sortino defines a risk adjusted performance ratio in which downside deviation is the risk measure.[10] In ex ante form, the Sortino Ratio is:

$$\mathrm{SortR}\left[\tilde{r}_P;\tau\right]=\frac{E\left[\tilde{r}_P\right]-\tau}{DD\left[\tilde{r}_P;\tau\right]} \tag{10}$$

As Figure 4 shows, the portfolio with the highest possible Sortino Ratio lies along the mean-downside deviation efficient frontier.

Figure 4: Mean-Downside Deviation Frontier and Portfolio with Maximum Sortino Ratio



James Knowles and I define a generalization of the Sortino Ratio that we call Kappa:[11]

$$K_n\left[\tilde{r}_P;\tau\right]=\frac{E\left[\tilde{r}_P;\tau\right]-\tau}{\sqrt[n]{LPM\left[\tilde{r}_P;\tau\right]}} \tag{11}$$

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[10] Sortino, Frank A., “From Alpha to Omega”, in *Managing Downside Risk in Financial Markets*, Frank A. Sortino and Stephen E. Satchell, eds., Reed Educational and Professional Publishing Ltd., 2001.

[11] Kaplan, Paul D. and James A. Knowles, “Kappa: A Generalized Downside Risk-Adjusted Performance Measure,” *Journal of Performance Measurement*, Spring 2004.

We show that the risk-adjusted performance measure defined by William Shadwick and Con Keating, Omega,[12] is simply a restatement of Kappa-1:

$$\Omega\left[\tilde{r}_P;\tau\right] = K_1\left[\tilde{r}_P;\tau\right] + 1 \tag{12}$$

In his 1959 book, Markowitz explored another form of semivariance, *below mean semivariance*:[13]

$$LPM_2^* = LPM_2\left(E\left[\tilde{r}_P\right]\right) \tag{13}$$

Although below mean semivarinance is not motivated by expected utility theory, it does embody the idea that it is only the left-hand of a return distribution that constitutes risk for an investor.

Value at Risk and Conditional Value at Risk

A risk measure that has become both popular and controversial is *Value at Risk* or *VaR*. Value at Risk is simply how much (or more) could be lost over a given period of time with a given probability. For example, if the 5% VaR of a portfolio is 12% for the upcoming 12 months, there is a 5% probability that 12 months from now, 12% or more of the portfolio's value will be lost. Mathematically, the 100pth VaR, $VaR\left[\tilde{r}_P;p\right]$ satisfies

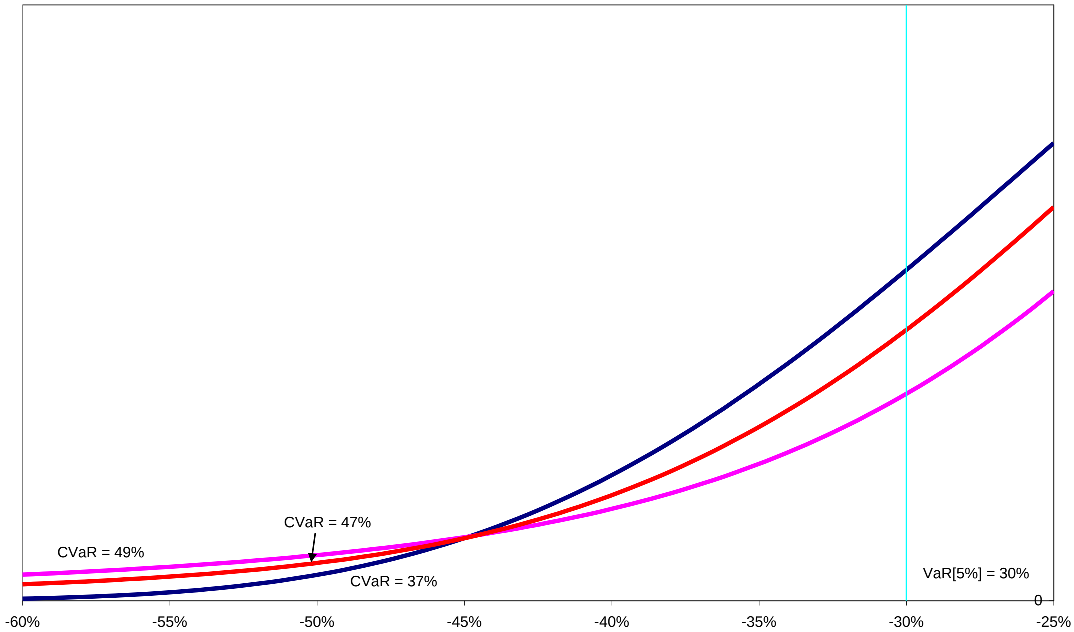
$$P\left[\tilde{r}_P \le -VaR\left[\tilde{r}_P;p\right]\right] = p \tag{14}$$

There are least two shortcomings that VaR has as a risk measure. Firstly, it is possible for a portfolio to have a VaR that is greater than the VaR of each of its constituents. That is, VaR violates the principle that diversification cannot increase risk. Secondly, it only indicates where the left tail of a distribution starts without indicating how much money could be lost should the VaR be breached. Figure 5 illustrates this point by showing the left tails of three distributions of returns that all have the same 5% VaR but have substantially different potential losses beyond the 5% VaR.

[12] Shadwick, William F. and Con Keating, "A Universal Performance Measure," *Journal of Performance Measurement*, Spring 2002.

[13] See note 6.

Figure 5: Left Tails of Distributions with Same VaRs and Different CVaRs



To overcome these shortcomings of Value at Risk, a related risk measure, *Conditional Value at Risk* or *CVaR* was created. Conditional Value at Risk is average loss show VaR be breached. Mathematically,

$$\mathrm{CVaR}\left[\tilde{r}_P;p\right] = -\mathrm{E}\left[\tilde{r}_P \mid \tilde{r}_P \le -\mathrm{VaR}\left[\tilde{r}_P;p\right]\right] \tag{15}$$

Since CVaR is the average of losses beyond VaR, CVaR ≥ VaR. The magnitude of the difference is the ratio of the 1st Lower Partial Moment to the given probability of loss:

$$\mathrm{CVaR}\left[\tilde{r}_P;p\right] - \mathrm{VaR}\left[\tilde{r}_P;p\right] = \frac{\mathrm{LPM}_1\left[\tilde{r}_P;-\mathrm{VaR}\left[\tilde{r}_P;p\right]\right]}{p} \tag{16}$$

Conclusions

Risk is a complicated and ambiguous concept so it is not surprising that there are a number of quantitative risk measures and measures of risk-adjusted performance. No single risk measure is perfect and in any application, it is wise to look at more than one.

In this primer, I have presented the theoretical motivations and formal definitions for a number of quantitative risk measures and in some cases, corresponding measures of risk-adjusted performance. I hope that this proves to be useful to those who encounter these measures in practice as to how to interpret them and understand both their strengths and their weaknesses.